Simple proof of the Prandtl-Meyer Integral

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- The Prandtl-Meyer integral:

$$I_{def} = \int_{M_1}^{M_2} \frac{\sqrt{M^2-1}}{\left(1+\frac{(\gamma-1)}{2}M^2\right)} \frac{dM}{M} =$$

$$= \left[\sqrt{\frac{\gamma+1}{\gamma-1}} \arctan\left(\sqrt{\frac{\gamma-1}{\gamma+1}(M^2-1)} \right) - \arctan\left(\sqrt{M^2-1}\right) \right]_{M_1}^{M_2}$$

- There is a radical in the numerator, from calculus we know that we should try to transform it. We will seek for a change of variables. If this fails we could try to multiply top and bottom by the same radical, but we will end with a radical in the denominator, which is not easy to handle (eventually we would use Euler substitution).
- There is also a polinomial in the bottom which configuration make us think in an arctangent as primitive.

- In a more general form:

$$I = \int \frac{\sqrt{x^m - 1}}{(1 + bx^m)} \frac{dx}{x}$$

- We can also make the change of variables:

$$\left.\begin{array}{c} I = \int \dfrac{\sqrt{x^m - 1}}{(1 + bx^m)} \dfrac{dx}{x} \\ x = e^u \Rightarrow dx = e^u du \Rightarrow dx = x du \end{array}\right\} \Rightarrow I = \int \frac{\sqrt{e^{mu} - 1}}{(1 + be^{mu})} du$$

- We could try to transform the integrand in a quotient of polinomials:

$$I = \int \frac{\sqrt{x^m - 1}}{(1 + bx^m)} \frac{dx}{x}$$

- “Renaming” the numerator as “Y”, we get:

$$\sqrt{x^m - 1} = y \Rightarrow x^m = y^2 + 1 \Rightarrow$$

$$\Rightarrow mx^{m-1}dx = 2ydy \Rightarrow mx^m x^{-1}dx = 2ydy \Rightarrow$$

$$\Rightarrow \frac{dx}{x} = \frac{2ydy}{mx^m} \Rightarrow \frac{dx}{x} = \frac{2ydy}{m(y^2+1)}$$

$$\Rightarrow I = \int \frac{y}{(1 + b(y^2 + 1))} \frac{2ydy}{m(y^2 + 1)}$$

- Reordering:

$$I=\int\frac{y}{(1+b(y^2+1))}\frac{2ydy}{m(y^2+1)}=\frac{2}{m}\int\frac{y^2dy}{(1+b+by^2)(y^2+1)}$$

- We can see that the denominator it is not completely factorized, but in this form we could expand the fraction so that can be integrated directly as arctangents. We make a Partial fraction expansion:

$$\frac{\mathrm{y}^2}{(1+\mathrm{b}+\mathrm{by}^2)(\mathrm{y}^2+1)}=\frac{\mathrm{A}}{(1+\mathrm{b}+\mathrm{by}^2)}+\frac{\mathrm{B}}{(\mathrm{y}^2+1)}\Rightarrow$$

$$\Rightarrow\frac{\mathrm{y}^2}{(1+\mathrm{b}+\mathrm{by}^2)(\mathrm{y}^2+1)}=\frac{\mathrm{A}\left(\mathrm{y}^2+1\right)+\mathrm{B}(1+\mathrm{b}+\mathrm{by}^2)}{(1+\mathrm{b}+\mathrm{by}^2)(\mathrm{y}^2+1)}\Rightarrow\mathrm{y}^2=(\mathrm{A}+\mathrm{bB})\,\mathrm{y}^2+(\mathrm{A}+\mathrm{bB}+\mathrm{B})\Rightarrow$$

$$\Rightarrow\left\{\left.\begin{array}{l}1=(\mathrm{A}+\mathrm{bB})\\ 0=(\mathrm{A}+\mathrm{bB}+\mathrm{B})\Rightarrow-\mathrm{B}=(\mathrm{A}+\mathrm{bB})\end{array}\right\}\right.\Rightarrow\left\{\begin{array}{l}A=(1+b)\\ \mathrm{B}=-1\end{array}\right.$$

- Substituting A, B and rearranging:

$$I = \frac{2}{m}\int\left(\frac{(1+\mathrm{b})}{(1+\mathrm{b}+\mathrm{by}^2)} - \frac{1}{(\mathrm{y}^2+1)}\right)dy = \frac{2}{m}\int\frac{(1+\mathrm{b})}{((1+\mathrm{b})+\mathrm{by}^2)}dy - \frac{2}{m}\int\frac{1}{(\mathrm{y}^2+1)}dy =$$

$$= \frac{2}{m}\int\frac{(1+\mathrm{b})}{(1+\mathrm{b})\left(1+\frac{\mathrm{by}^2}{(1+\mathrm{b})}\right)}dy - \frac{2}{m}\int\frac{1}{(1+\mathrm{y}^2)}dy =$$

$$= \frac{2}{\mathrm{m}}\frac{\sqrt{\mathrm{b}+1}}{\sqrt{\mathrm{b}+1}}\cdot\frac{\sqrt{\mathrm{b}}}{\sqrt{\mathrm{b}}}\int\frac{1}{\left(1+\left(\frac{\sqrt{\mathrm{b}}\cdot\mathrm{y}}{\sqrt{\mathrm{b}+1}}\right)^2\right)}\mathrm{dy} - \frac{2}{\mathrm{m}}\int\frac{1}{(1+\mathrm{y}^2)}\mathrm{dy} =$$

$$= \frac{2}{\mathrm{m}}\frac{\sqrt{\mathrm{b}+1}}{\sqrt{\mathrm{b}}}\int\frac{\frac{\sqrt{\mathrm{b}}}{\sqrt{\mathrm{b}+1}}}{\left(1+\left(\frac{\sqrt{\mathrm{b}}\cdot\mathrm{y}}{\sqrt{\mathrm{b}+1}}\right)^2\right)}\mathrm{dy} - \frac{2}{m}\int\frac{1}{(1+\mathrm{y}^2)}dy$$

- We perform the integration and substitute terms:

$$\left.\begin{aligned}
&\mathrm{I} = \frac{2}{\mathrm{m}}\frac{\sqrt{\mathrm{b}+1}}{\sqrt{\mathrm{b}}}\arctan\left(\frac{\sqrt{\mathrm{b}}\cdot \mathrm{y}}{\sqrt{\mathrm{b}+1}}\right) - \frac{2}{\mathrm{m}}\arctan(\mathrm{y}) \\
&\mathrm{y} = \sqrt{\mathrm{x}^{\mathrm{m}}-1}
\end{aligned}\right\} \Rightarrow$$

$$\left.\begin{aligned}
&\Rightarrow \mathrm{I} = \frac{2}{\mathrm{m}}\frac{\sqrt{\mathrm{b}+1}}{\sqrt{\mathrm{b}}}\arctan\left(\frac{\sqrt{\mathrm{b}}\sqrt{\mathrm{x}^{\mathrm{m}}-1}}{\sqrt{\mathrm{b}+1}}\right) - \frac{2}{\mathrm{m}}\arctan\left(\sqrt{\mathrm{x}^{\mathrm{m}}-1}\right) \\
&\left\{M = x, \mathrm{b} = \frac{\gamma-1}{2}, m = 2\right\}
\end{aligned}\right\} \Rightarrow$$

$$\Rightarrow \mathrm{I} = \frac{2}{2}\frac{\sqrt{\frac{\gamma-1}{2}+1}}{\sqrt{\frac{\gamma-1}{2}}}\arctan\left(\sqrt{\frac{2(\gamma-1)(\mathrm{M}^2-1)}{2(\gamma-1+2)}}\right) - \frac{2}{2}\arctan\left(\sqrt{M^2-1}\right)$$

- Finally using Barrow's rule:

$$I_{def} = \int_{M_1}^{M_2} \frac{\sqrt{M^2-1}}{\left(1+\frac{(\gamma-1)}{2}M^2\right)}\frac{dM}{M} =$$

$$= \left[\sqrt{\frac{\gamma+1}{\gamma-1}}\arctan\left(\sqrt{\frac{\gamma-1}{\gamma+1}(M^2-1)}\right) - \arctan\left(\sqrt{M^2-1}\right)\right]_{M_1}^{M_2} =$$

$$= \left[\sqrt{\frac{\gamma+1}{\gamma-1}}\arctan\left(\sqrt{\frac{\gamma-1}{\gamma+1}({M_2}^2-1)}\right) - \arctan\left(\sqrt{{M_2}^2-1}\right)\right] -$$

$$- \left[\sqrt{\frac{\gamma+1}{\gamma-1}}\arctan\left(\sqrt{\frac{\gamma-1}{\gamma+1}({M_1}^2-1)}\right) - \arctan\left(\sqrt{{M_1}^2-1}\right)\right]$$